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Exercises for 'Functional Analysis 2' [MATH-404]

(12/05/2025)

Ex 11.1 (Second order optimality conditions for local minimizers*)
Let X be a Banach space and $U \subset X$ be open. Consider a twice-differentiable function $F : U \to \mathbb{R}$ and $x_0 \in U$.

a) Show that when x_0 is a local minimizer of F, then $F''(x_0)(h,h) \geq 0$ for all $h \in X$.
b) Assume that $F'(x_0) = 0$ and that there exists $\delta > 0$ such that $F''(x_0)(h,h) \geq \delta \|h\|^2$ for all $h \in X$, then x_0 is an isolated local minimizer of F.

Ex 11.2 (Characterization of convexity)
Let $U \subset X$ be an open and convex subset of a Banach space X, and let $F \colon U \to \mathbb{R}$ be Gâteaux-differentiable. Show that the following are equivalent :

a) F is convex.
b) $F(y) \geq F(x) + \delta F(x)(y-x)$ for every $x, y \in U$.
c) $(\delta F(x) - \delta F(y))(x-y) \geq 0$ for every $x, y \in U$.

Hint: For c) $\Rightarrow$ a) study the function $[0,1] \ni \lambda \mapsto \phi(\lambda) = F(\lambda x + (1-\lambda)y) - \lambda F(x) - (1-\lambda)F(y)$.

Ex 11.3 (Newton's method)
Suppose that $F \colon X \to X$ is a C^2 operator on a Banach space X with F'' bounded and $F(x^*) = 0$ for some $x^* \in X$. Suppose moreover that $F'(x^*)^{-1}$ exists as a continuous linear operator on X.

a) Show that there is $\delta > 0$ such that for all $x \in B(x^*, \delta)$, $S(x) := F'(x)^{-1}$ exists as a continuous linear operator on X and $\sup_{x \in B(x^*,\delta)} \|S(x)\|_{\mathcal{L}(X)} < +\infty$.

   Hint: Note that $F'(x) = F'(x^*)\big[I + S(x^*)\big(F'(x) - F'(x^*)\big)\big]$ and use the properties of the Neumann series.

Set

$$T(x) = x - \big[F'(x)^{-1}\big]F(x), \quad x \in B(x^*, \delta).$$

b) Show that there is $0 < \hat{\delta} < \delta$ such that $T \colon B(x^*, \hat{\delta}) \to B(x^*, \hat{\delta})$.

   Hint: Use the Taylor expansion to estimate $\|F(x) - F(x^*) - F'(x^*)(x - x^*)\|_X$.

Consider a sequence

$$x_{n+1} = T(x_n) = x_n - \big[F'(x_n)^{-1}\big]F(x_n), \quad x_0 \in B(x^*, \hat{\delta}).$$

c) Show that there is a constant $M > 0$ such that

$$\|x_n - x^*\|_X \leq \frac{(M\|x_0 - x^*\|_X)^{2^n}}{M},$$

   thus if we have a 'good' initial guess x_0, i.e., $\|x_0 - x^*\|_X < \hat{\delta}$, we have convergence $x_n \to x^*$ of quadratic order.